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General technique to produce isochronous Hamiltonians

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Abstract
We introduce a new technique—characterized by an *arbitrary* positive constant Ω, with which we associate the period $T = 2\pi/\Omega$—to 'Ω-modify' a Hamiltonian so that the new Hamiltonian thereby obtained is *entirely isochronous*, namely it yields motions *all* of which (except possibly for a lower dimensional set of *singular* motions) are *periodic* with the *same fixed* period T in *all* their degrees of freedom. This technique transforms *real autonomous* Hamiltonians into Ω-modified Hamiltonians which are also *real* and *autonomous*, and it is widely applicable, for instance, to the most general many-body problem characterized by Newtonian equations of motion ('acceleration equal force') provided it is *translation invariant*. The Ω-modified Hamiltonians are of course *not* translation invariant, but for $\Omega = 0$ they reduce (up to marginal changes) to the unmodified Hamiltonians they were obtained from. Hence, when this technique is applied to translation-invariant Hamiltonians yielding, in their center-of-mass systems, *chaotic* motions with a natural time scale much smaller than T, the corresponding Ω-modified Hamiltonians shall display a *chaotic* behavior for quite some time before the *isochronous* character of the motions takes over. We moreover show that the quantized versions of these Ω-modified Hamiltonians feature *equispaced* spectra.

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1. Introduction

Recently, much research (see, for instance, [1–4]) has been devoted to the identification and investigation of dynamical systems which are *isochronous*, featuring an *open* (hence fully dimensional) region in their phase space in which *all* their motions are *completely periodic* (namely, periodic in *all* their degrees of freedom) with the *same* fixed period: for a review of these developments, see [5, 6]. In this paper, we introduce a new technique characterized by an *arbitrary* positive constant Ω. It allows us to 'Ω-modify' a Hamiltonian so that the new Hamiltonian thereby obtained is *entirely isochronous*, yielding motions *all* of which (except possibly for a lower dimensional set of *singular* motions) are *periodic* with the *same* fixed period

$$T = \frac{2\pi}{\Omega} \tag{1}$$

in *all* their degrees of freedom. It transforms *real autonomous* Hamiltonians into Ω-modified Hamiltonians which are also *real* and *autonomous*. It is widely applicable (as shown below), but for definiteness we focus hereafter mainly on the—most general, except for the requirement that it be *translation invariant*—nonrelativistic many-body problem, characterized by Newtonian equations of motion ('acceleration equal force'). The Ω-modified Hamiltonian thereby obtained is of course *not* translation invariant, but for $\Omega = 0$ it reduces (up to marginal changes, see below) to the unmodified Hamiltonian it was obtained from. Hence, when this technique is applied to a translation-invariant many-body problem yielding, in its center-of-mass system, *chaotic* motions with a natural time scale T_C, it is possible to manufacture—via an assignment of the constant Ω entailing that the corresponding period T, see (1), is much larger than this time scale, $T \gg T_C$—an Ω-modified many-body problem that shall display some kind of *chaotic* behavior for quite some time before the *isochronous* character of all its motions takes over.

In section 2, we report the main findings of this paper: the new technique of Ω-modification and its properties, including a terse discussion of the *transient* '*chaotic*' phenomenology outlined above and an illustration of our findings via a very simple, quite explicit example. In section 3 we provide a terse treatment of these Ω-modified Hamiltonians in the *quantal* context, demonstrating in particular their property to feature an *equispaced* spectrum with spacing $\hbar\Omega$—each eigenvalue of which is of course generally *infinitely degenerate*. A terse section 4 entitled 'outlook' concludes the paper.

2. Results

In this section 2 we report our results, after a preliminary introduction largely devoted to establishing our notation. We mainly focus on the Hamiltonian characterizing the standard nonrelativistic N-body problem—restricting attention for simplicity firstly to the one-dimensional case with equal masses and outlining only later the extension to more general Hamiltonians.

2.1. Preliminaries and notation

We write as follows (in self-evident notation) the (simplest version of the) Hamiltonian characterizing the standard nonrelativistic N-body problem:

$$H(\underline{p}, \underline{q}) = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + V(\underline{q}). \tag{2}$$

Notation. N is an arbitrary positive integer ($N \geqslant 2$), and hereafter the index n runs from 1 to N unless otherwise indicated. For simplicity we set throughout the mass of the particles to unity (and dimensions are supposed to be accordingly adjusted); the canonical coordinates q_n and momenta p_n are the N components of the N-vectors $\underline{q}$ and $\underline{p}$ respectively, and the many-body potential energy function $V(\underline{q})$ is essentially arbitrary, except for the requirement of *translation invariance*,

$$V(\underline{q}+\underline{a}) = V(\underline{q}). \tag{3a}$$

In this formula, $\underline{a}$ denotes an *arbitrary* constant N-vector; for infinitesimal $\underline{a}$, the corresponding formula reads, of course,

$$\sum_{n=1}^{N} \frac{\partial V(\underline{q})}{\partial q_n} = 0. \tag{3b}$$

We hereafter denote with P the total momentum and with Q the (canonically conjugate) center-of-mass coordinate:

$$P = \sum_{n=1}^{N} p_n, \qquad Q = \frac{1}{N}\sum_{n=1}^{N} q_n. \tag{4}$$

Thanks to the translation invariance property (3b),

$$[H, P] = 0. \tag{5}$$

Here and hereafter, the Poisson bracket $[F, G]$ of two functions $F(\underline{p}, \underline{q})$ and $G(\underline{p}, \underline{q})$ of the canonical variables is defined as follows:

$$[F, G] \equiv \sum_{n=1}^{N} \left[\frac{\partial F(\underline{p}, \underline{q})}{\partial p_n} \frac{\partial G(\underline{p}, \underline{q})}{\partial q_n} - \frac{\partial G(\underline{p}, \underline{q})}{\partial p_n} \frac{\partial F(\underline{p}, \underline{q})}{\partial q_n} \right]. \tag{6}$$

Let us recall that the evolution of any function $F(\underline{p}, \underline{q})$ of the canonical coordinates $\underline{p}, \underline{q}$ is determined by the equation

$$F' = [H, F], \tag{7}$$

where the appended prime denotes differentiation with respect to the 'timelike' variable corresponding to the evolution induced by the Hamiltonian H.

It is now convenient to introduce the 'relative coordinates' x_n and the 'relative momenta' y_n via the definitions

$$x_n = q_n - Q, \qquad y_n = p_n - \frac{P}{N}. \tag{8a}$$

Note that these are *not* canonically conjugated quantities, since $[y_n, x_m] = \delta_{nm} - 1/N$, and they are *not* independent since they obviously satisfy the property

$$\sum_{n=1}^{N} x_n = \sum_{n=1}^{N} y_n = 0. \tag{8b}$$

It is moreover convenient to introduce the relative-motion Hamiltonian $h(\underline{y}, \underline{x})$ via the formula

$$h(\underline{y}, \underline{x}) = \frac{1}{2}\sum_{n=1}^{N} y_n^2 + V(\underline{x}) = \frac{1}{4N}\sum_{n,m=1}^{N} (p_n - p_m)^2 + V(\underline{q}), \tag{9a}$$

so that

$$H(\underline{p}, \underline{q}) = \frac{P^2}{2N} + h(\underline{y}, \underline{x}). \tag{9b}$$

Note that to write the two versions of (9a), we used the identification $V(\underline{x}) = V(\underline{q})$ implied by (8a) and (3a). Note moreover that this definition, (9a), of the relative-motion Hamiltonian $h(\underline{y}, \underline{x})$ entails that it Poisson-commutes with both P and Q:

$$[P, h] = [Q, h] = 0. \tag{9c}$$

For completeness and future reference, let us also display the equations of motion implied by the Hamiltonian $H(\underline{p}, \underline{q})$, see (2):

$$q_n' = p_n, \qquad p_n' = -\frac{\partial V(\underline{q})}{\partial q_n}, \qquad q_n'' = -\frac{\partial V(\underline{q})}{\partial q_n}, \tag{10a}$$

where (for reasons that will be clear below) we denote as τ the independent variable corresponding to this Hamiltonian flow and with appended primes the differentiations with respect to these variable:

$$q_n \equiv q_n(\tau), \qquad p_n \equiv p_n(\tau); \qquad q_n' \equiv \frac{\partial q_n(\tau)}{\partial \tau}, \qquad p_n' \equiv \frac{\partial p_n(\tau)}{\partial \tau}. \tag{10b}$$

Hence, from (4) and (3b),

$$Q' = \frac{P}{N}, \qquad P' = 0, \tag{11a}$$

yielding

$$Q(\tau) = Q(0) + \frac{P(0)}{N}\tau, \qquad P(\tau) = P(0), \tag{11b}$$

and from (8a) and (10a),

$$x_n' = y_n = \frac{\partial h(\underline{y}, \underline{x})}{\partial y_n}, \qquad y_n' = -\frac{\partial V(\underline{x})}{\partial x_n} = -\frac{\partial h(\underline{y}, \underline{x})}{\partial x_n}. \tag{12}$$

Note that these equations have the standard Hamiltonian form even though, as mentioned above, x_n and y_n are *not* canonically conjugated variables.

2.2. *The isochronous Hamiltonian*

The Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$ is now defined by the formula

$$\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega) = \frac{1}{2}\left\{\left[\tilde{P} + \frac{\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{b}\right]^2 + \Omega^2 \tilde{Q}^2\right\}, \tag{13a}$$

where b is an *arbitrary* constant (introduced for dimensional reasons: it has the dimensions of a momentum), Ω is of course a *positive* constant and we introduced (perhaps unnecessarily) the notation

$$\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}}) \equiv h(\underline{\tilde{y}}, \underline{\tilde{x}}) = \frac{1}{2}\sum_{n=1}^{N} \tilde{y}_n^2 + V(\underline{\tilde{x}}). \tag{13b}$$

Here the superimposed tildes over the canonical coordinates and momenta, $\tilde{q}_n \equiv \tilde{q}_n(t)$ and $\tilde{p}_n \equiv \tilde{p}_n(t)$, are introduced to emphasize that these variables evolve now according to the Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$. Likewise the total momentum $\tilde{P} \equiv \tilde{P}(t)$ and the center-of-mass coordinate $\tilde{Q} \equiv \tilde{Q}(t)$, as well as the relative coordinates and momenta $\tilde{x}_n \equiv \tilde{x}_n(t)$ and $\tilde{y}_n \equiv \tilde{y}_n(t)$, are now defined by formulae analogous to the previous ones:

$$\tilde{P} = \sum_{n=1}^{N} \tilde{p}_n, \qquad \tilde{Q} = \frac{1}{N}\sum_{n=1}^{N} \tilde{q}_n, \tag{14a}$$

$$\tilde{x}_n = \tilde{q}_n - \tilde{Q}, \qquad \tilde{y}_n = \tilde{p}_n - \frac{\tilde{P}}{N}, \tag{14b}$$

and, as already indicated above, the corresponding independent variable is now denoted as t ('time'), and differentiations with respect to this variable will be denoted, as usual, by superimposed dots.

It is now easily seen that they hold the following Poisson commutation formulae:

$$[\tilde{H}, \tilde{Q}] = \tilde{P} + \frac{\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{b}, \qquad [\tilde{H}, \tilde{P}] = -\Omega^2 \tilde{Q}, \qquad [\tilde{H}, \tilde{h}] = 0, \tag{15}$$

so that the quantities $\tilde{Q}$, $\tilde{P}$ and $\tilde{h}$ evolve as follows under the flow induced by the Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$, see (13):

$$\dot{\tilde{Q}} = \tilde{P} + \frac{\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{b}, \qquad \dot{\tilde{P}} = -\Omega^2 \tilde{Q}, \qquad \dot{\tilde{h}} = 0, \tag{16}$$

entailing

$$\tilde{Q}(t) = \tilde{Q}(0)\cos(\Omega t) + \dot{\tilde{Q}}(0)\frac{\sin(\Omega t)}{\Omega}, \tag{17a}$$

$$\tilde{P}(t) = \tilde{P}(0)\cos(\Omega t) + \dot{\tilde{P}}(0)\frac{\sin(\Omega t)}{\Omega} + \frac{\tilde{h}[\underline{\tilde{y}}(0), \underline{\tilde{x}}(0)]}{b}[\cos(\Omega t) - 1], \tag{17b}$$

$$\tilde{h}[\underline{\tilde{y}}(t), \underline{\tilde{x}}(t)] = \tilde{h}[\underline{\tilde{y}}(0), \underline{\tilde{x}}(0)]. \tag{17c}$$

It is moreover plain that the total momentum $\tilde{P}$ and the center-of-mass coordinate $\tilde{Q}$ Poisson-commute with the relative-motion momenta and coordinates $\tilde{y}_n$ and $\tilde{x}_n$ hence also with any function of these variables,

$$[\tilde{P}, \tilde{x}_n] = [\tilde{P}, \tilde{y}_n] = [\tilde{P}, \tilde{h}] = [\tilde{Q}, \tilde{x}_n] = [\tilde{Q}, \tilde{y}_n] = [\tilde{Q}, \tilde{h}] = 0; \tag{18}$$

hence, the evolution equations of the relative-motion coordinates and momenta $\tilde{x}_n$ and $\tilde{y}_n$ under the flow induced by the Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$ read

$$\dot{\tilde{x}}_n = \frac{1}{b}\left[\tilde{P} + \frac{\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{b}\right]\frac{\partial \tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{\partial \tilde{y}_n} = \frac{\dot{\tilde{Q}}}{b}\frac{\partial \tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{\partial \tilde{y}_n}, \tag{19a}$$

$$\dot{\tilde{y}}_n = -\frac{1}{b}\left[\tilde{P} + \frac{\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{b}\right]\frac{\partial \tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{\partial \tilde{x}_n} = -\frac{\dot{\tilde{Q}}}{b}\frac{\partial \tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{\partial \tilde{x}_n}, \tag{19b}$$

namely, via (17) and (19),

$$\dot{\tilde{x}}_n = C\cos[\Omega(t - t_0)]\frac{\partial \tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{\partial \tilde{y}_n}, \tag{20a}$$

$$\dot{\tilde{y}}_n = -C\cos[\Omega(t - t_0)]\frac{\partial \tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})}{\partial \tilde{x}_n}, \tag{20b}$$

where

$$C = \frac{\sqrt{2\tilde{H}}}{b}, \qquad \sin(\Omega t_0) = -\frac{\Omega \tilde{Q}(0)}{bC} = -\frac{\Omega \tilde{Q}(0)}{\sqrt{2\tilde{H}}}. \tag{20c}$$

It is now crucial to observe—by comparing these evolution equations with (12)—that it is justified to set

$$\tilde{x}_n(t) = x_n(\tilde{t}), \qquad \tilde{y}_n(t) = y_n(\tilde{t}), \tag{21a}$$

with

$$\tilde{t} = C\frac{\sin[\Omega(t - t_0)]}{\Omega} = \frac{\tilde{Q}(t)}{b}. \tag{21b}$$

Here, t_0 is defined mod$(2\pi/\Omega)$ by (20c); the coordinates and momenta $x_n(\tilde{t})$ and $y_n(\tilde{t})$ evolve as functions of $\tilde{t}$ according to the original flow yielded by the Hamiltonian $H(\underline{p}, \underline{q})$, see (12), and are uniquely identified by 'initial' data assigned at

$$\tilde{t}_0 = \frac{\tilde{Q}(0)}{b} = \frac{C\tilde{Q}(0)}{\sqrt{2\tilde{H}}}, \tag{21c}$$

according to the following prescription:

$$x_n(\tilde{t}_0) = \tilde{x}_n(0), \qquad y_n(\tilde{t}_0) = \tilde{y}_n(0), \tag{21d}$$

where $\tilde{x}_n(0)$ and $\tilde{y}_n(0)$ are the initial data for the relative-motion dynamics induced by the Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$. In this manner, the dynamics of the canonical coordinate and momenta $\tilde{q}_n(t)$ and $\tilde{p}_n(t)$ evolving according to our Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$, see (13), is finally obtained via (14b) and (17) respectively. It is now plain that this dynamics is *isochronous* with period T, see (14b), (17) and (21); note that the time evolution of the coordinates $x_n(\tilde{t})$ and $y_n(\tilde{t})$ is uniquely well defined for all *real* time, see (12), unless it runs into singularities, which should not be the case for physically sound models, and in any case should only happen exceptionally.

2.3. *The $\Omega = 0$ limit and the behavior of the isochronous system over time scales much shorter than T*

It is plain that, when Ω vanishes, the Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$, see (13), does not quite reduce to the unmodified Hamiltonian $H(\underline{p}, \underline{q})$, see (2), but it is also clear that the dynamics yielded by the Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; 0)$ differs only marginally from that yielded by the original Hamiltonian $H(\underline{p}, \underline{q})$. To illustrate this point we now display the version that the most relevant formulae written above take when Ω vanishes, $\Omega = 0$. Let us consider firstly the evolution (see (17) and (16)) of the center of mass $\tilde{Q}$ and the total momentum $\tilde{P}$:

$$\tilde{Q}(t) = \tilde{Q}(0) + \left[\tilde{P}(0) + \frac{\tilde{h}[\underline{\tilde{y}}(0), \underline{\tilde{x}}(0)]}{b}\right]t, \qquad \tilde{P}(t) = \tilde{P}(0), \tag{22a}$$

to be compared with the analogous evolution yielded by the Hamiltonian $H(\underline{p}, \underline{q})$,

$$Q(t) = Q(0) + \frac{P(0)}{N}t, \qquad P(t) = P(0). \tag{22b}$$

Next, let us compare the evolution of the relative-motion variables $\tilde{x}_n(t)$ and $\tilde{y}_n(t)$. The relevant formula is of course always (21a), but now with (21b) replaced by (see (20c))

$$\tilde{t} = Ct + \frac{\tilde{Q}(0)}{b}. \tag{23}$$

These formulae confirm the assertion that the dynamics yielded by the Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; 0)$ differs only marginally from that yielded by the original Hamiltonian $H(\underline{p}, \underline{q})$.

In fact an analogous relationship—entailing that $\tilde{t}$ on a sufficiently short time scale varies linearly in t—holds generally, since in the neighborhood of any time $\bar{t}$—except when $\dot{\tilde{Q}}(\bar{t})$ vanishes—

$$\tilde{t}(t) = \frac{\tilde{Q}(\bar{t}) + \dot{\tilde{Q}}(\bar{t})(t - \bar{t})}{b} + O[(t - \bar{t})^2]. \tag{24}$$

One therefore finds that throughout the time evolution, the Ω-modified dynamics differs from the unmodified one solely by a time rescaling—by a possibly negative coefficient—and by a time shift. The coefficient and the shift are time independent over a time scale much smaller than the *isochrony* period $T = 2\pi/\Omega$, but vary periodically with period T, see (21*b*). A peculiar state of affairs arises, however, whenever $\dot{\tilde{Q}}(t)$ vanishes, namely when $\mathrm{d}\tilde{t}/\mathrm{d}t$ changes its sign: this of course happens twice within every time period T, see (21*b*), this being in fact a consequence of the periodicity of $\tilde{t}(t)$, which itself is the cause of the *isochrony*. These aspects are apparent in figures 1 and 2.

2.4. Transient chaos

It is interesting to speculate on the application of this Ω-modification technique to any Hamiltonian describing a translation-invariant many-body problem featuring, in its center-of-mass system, *chaotic* motions with a natural time scale T_C. Then—provided that the constant Ω is assigned so that the *isochrony* period T, see (1), is much larger than this time scale, $T \gg T_C$—the Ω-modified problem shall exhibit some kind of *chaotic* behavior for quite some time before the *isochronous* character of all its motions takes over, causing thereafter a recurrent evolution. This phenomenology—qualitative rather than quantitative as it necessarily is, since a *precise* definition of *chaoticity* requires generally that a system displaying it be observed for *infinite* time—is nevertheless quite remarkable. A more detailed analysis of this phenomenology—including numerical investigations—is postponed to a separate paper.

2.5. More general Hamiltonians

All the formulae written above are appropriate to the one-dimensional N-body problem with equal mass particles. These restrictions have been imposed merely for the sake of simplicity: the alert reader will have no difficulty to extend the treatment—if need be—to many-body problems with particles having different masses and moving in higher dimensional space. Of course, in such a case, there is an ample choice of the collective variables playing a role analogous to that played above by the total momentum and the center-of-mass coordinate; for instance, in a multidimensional context, any component of the vectors $\vec{P}$ and $\vec{Q}$ representing the total momentum and the center-of-mass coordinate could be used, or, not to spoil rotation invariance, one could replace, in the definition (13*a*) of the Ω-modified Hamiltonian, P with $\vec{P} \cdot \vec{Q}/\sqrt{\vec{Q} \cdot \vec{Q}}$ and Q^2 with $\vec{Q} \cdot \vec{Q}$.

We now show that the technique of Ω-modification can be applied to quite general Hamiltonians $H(\underline{p}, \underline{q})$, provided they allow the definition of a collective quantity $\Theta(\underline{p}, \underline{q})$ satisfying the Poisson bracket formula

$$[H(\underline{p}, \underline{q}), \Theta(\underline{p}, \underline{q})] = 1, \tag{25a}$$

so that this variable $\Theta(\underline{p}, \underline{q})$ is canonically conjugated to the Hamiltonian $H(\underline{p}, \underline{q})$ and therefore evolves, under the flow induced by this Hamiltonian—a flow we find convenient to characterize, as above, via the independent variable τ—according to the formula

$$\Theta' = 1, \tag{25b}$$

where the appended prime denotes of course total differentiation with respect to this independent variable τ (performed via the dependence from this variable τ of the canonical variables $\underline{p}$ and $\underline{q}$). Note that the formula (25*a*) identifies $\Theta(\underline{p}, \underline{q})$ up to the addition of any function of the canonical variables that Poisson-commutes with the Hamiltonian $H(\underline{p}, \underline{q})$, namely that it is a constant of motion under the flow induced by this Hamiltonian. Then an Ω-modified Hamiltonian yielding *entirely isochronous* motions—or at least entailing that $2N-1$ functionally independent functions of the canonical variables $q_n(t)$ and $p_n(t)$ *all* evolve *periodically* with period $T = 2\pi/\Omega$; see below—reads

$$\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega) = \tfrac{1}{2}\{B^{-2}[H(\underline{\tilde{p}}, \underline{\tilde{q}})]^2 + B^2\Omega^2[\Theta(\underline{\tilde{p}}, \underline{\tilde{q}})]^2\}, \tag{25c}$$

provided the resulting dynamics does not run into a singularity within a time $T = 2\pi/\Omega$. The constant B is introduced here to adjust the dimensions; to do so it must of course be, dimensionally, a momentum.

An example of the *effectiveness* of this rule is provided by our previous treatment, corresponding to the assignment $H = B(P + h/b)$ and $\Theta = Q/B$, see (13*a*) and (25*c*). On the other hand, an example of *ineffective* assignment is provided by the one degree-of-freedom Ω-modified Hamiltonian

$$\tilde{H}(\tilde{p}, \tilde{q}; \Omega) = \frac{1}{2}\left(\frac{\tilde{p}^4}{4B^2} + \frac{B^2\Omega^2\tilde{q}^2}{\tilde{p}^2}\right), \tag{26a}$$

which corresponds to (25*c*) with

$$H(\underline{p}, \underline{q}) = \frac{p^2}{2}, \qquad \Theta(p, q) = \frac{q}{p}. \tag{26b}$$

Indeed this Ω-modified Hamiltonian, (26*a*), is easily seen to yield the equation of motion

$$\ddot{H} + \Omega^2 H = 0, \tag{26c}$$

entailing

$$H(t) = H(0)\cos(\Omega t) + \dot{H}(0)\frac{\sin(\Omega t)}{\Omega} = A\cos(\Omega t + \alpha), \tag{26d}$$

which is clearly incompatible with the positivity of H, see its definition (26*b*).

To our knowledge, there do not exist general criteria for the absence of such singularities. This potential difficulty should therefore be investigated on a case-by-case basis.

To prove the result stated above, we make the following elementary remark: through each phase-space point $\underline{p}, \underline{q}$ goes one, and only one, orbit of $H(\underline{p}, \underline{q})$. Due to the basic property of Θ to always increase linearly in τ along all orbits of H in phase space (see (25*a*)), there is one and only one point $\underline{\eta}(\underline{p}, \underline{q}), \underline{\xi}(\underline{p}, \underline{q})$ on that orbit such that

$$\Theta(\underline{\eta}, \underline{\xi}) = 0. \tag{27}$$

Hence to every phase-space point $\underline{p}, \underline{q}$ (identified by $2N$ scalar coordinates) there correspond *uniquely* the quantities $\underline{\eta}(\underline{p}, \underline{q}), \underline{\xi}(\underline{p}, \underline{q})$ and $\Theta(\underline{p}, \underline{q})$, and the converse is also true, namely to assigned values (in phase space) of the quantities $\underline{\eta}, \underline{\xi}$ and Θ—with the latter satisfying the constraint (27)—there correspond *uniquely* quantities $\underline{p}, \underline{q}$ (these amount to $2N$ scalar coordinates, while the $2N + 1$ quantities $\underline{\eta}, \underline{\xi}$ and Θ are constrained by the single scalar condition (27); thus the numbers of independent scalar coordinates match). Hence in the following we can consider at our convenience, as dependent variables evolving under the relevant dynamics, either the variables $\underline{p}, \underline{q}$ or the variables $\underline{\eta}, \underline{\xi}$ and Θ, on the understanding that there exists a one-to-one transformation among these two sets of variables. We now note that, since the phase-space point $\underline{\eta}, \underline{\xi}$ obviously does not vary as $\underline{p} = \underline{p}(\tau), \underline{q} = \underline{q}(\tau)$ go through the orbit, these quantities $\underline{\eta}$ and $\underline{\xi}$ are in fact integrals of the motion induced by the Hamiltonian H; hence they Poisson-commute with H,

$$[H, \underline{\eta}] = [H, \underline{\xi}] = 0. \tag{28}$$

Furthermore, one observes that there holds the Poisson-commutation relations

$$[H(\underline{p}, \underline{q}), [\underline{\xi}(\underline{p}, \underline{q}), \Theta(\underline{p}, \underline{q})]] = [H(\underline{p}, \underline{q}), [\underline{\eta}(\underline{p}, \underline{q}), \Theta(\underline{p}, \underline{q})]] = 0, \tag{29}$$

which follow from the observation that the other two terms in the corresponding Jacobi identities vanish trivially, thanks to (25*a*) and (28). This suggests introducing the quantities

$$\underline{F}(\underline{\eta},\underline{\xi}) = [\underline{\xi}(\underline{p},\underline{q}),\Theta(\underline{p},\underline{q})], \qquad \underline{G}(\underline{\eta},\underline{\xi}) = [\underline{\eta}(\underline{p},\underline{q}),\Theta(\underline{p},\underline{q})], \tag{30a}$$

which then obviously Poisson-commute with H (see (29)):

$$[F,H] = [G,H] = 0. \tag{30b}$$

Note that this fact entails that, as already anticipated by the notation used above, these quantities depend on the coordinates $\underline{\xi}$ and $\underline{\eta}$, which indeed Poisson-commute with H, see (28), but are independent of Θ, which does *not* commute with H, see (25*a*). This fact plays a crucial role, see below.

Let us now examine the dynamics induced by the Ω-modified Hamiltonian $\tilde{H}$, see (25*c*). Clearly under this evolution—now characterized by the independent variable t ('time')—the quantities H and Θ evolve respectively as the momentum and coordinate of a one-dimensional standard harmonic oscillator with circular frequency Ω. Indeed, (25*c*) and (25*a*) entail

$$\dot{\Theta} = [\tilde{H},\Theta] = B^{-2}H, \qquad \dot{H} = [\tilde{H},H] = -B^2\Omega^2\Theta; \tag{31a}$$

hence

$$\ddot{\Theta} + \Omega^2\Theta = \ddot{H} + \Omega^2 H = 0, \tag{31b}$$

and hence

$$\Theta(t) = \Theta(0)\cos(\Omega t) + \frac{H(0)\sin(\Omega t)}{B^2\Omega} = A\cos[\Omega(t-t_0)], \tag{32a}$$

$$H(t) = H(0)\cos(\Omega t) - B^2\Theta(0)\Omega\sin(\Omega t) = AB^2\Omega\sin[\Omega(t-t_0)], \tag{32b}$$

$$A^2 = [\Theta(0)]^2 + \left[\frac{H(0)}{B^2\Omega}\right]^2, \qquad \sin(\omega t_0) = -\frac{H(0)}{AB^2\Omega}. \tag{32c}$$

It is therefore plain that these quantities evolve now periodically, with period $T = 2\pi/\Omega$.

To characterize the rest of the dynamics induced by the Ω-modified Hamiltonian $\tilde{H}$, there remains to investigate the motion of the coordinates $\underline{\tilde{\xi}}$ and $\underline{\tilde{\eta}}$, where the superimposed tildes have now been added to emphasize that these quantities evolve now according to the flow induced by the Ω-modified Hamiltonian $\tilde{H}$. The relevant equations of motion read (see (25*c*) and (28), as well as (31*a*) and (32*a*))

$$\dot{\underline{\tilde{\xi}}} = -B^2\Omega^2\Theta\underline{F}(\underline{\tilde{\eta}},\underline{\tilde{\xi}}) = \dot{H}\underline{F}(\underline{\tilde{\eta}},\underline{\tilde{\xi}}) = -AB^2\Omega^2\cos[\Omega(t-t_0)]\underline{F}(\underline{\tilde{\eta}},\underline{\tilde{\xi}}), \tag{33a}$$

$$\dot{\underline{\tilde{\eta}}} = -B^2\Omega^2\Theta\underline{G}(\underline{\tilde{\eta}},\underline{\tilde{\xi}}) = \dot{H}\underline{G}(\underline{\tilde{\eta}},\underline{\tilde{\xi}}) = -AB^2\Omega^2\cos[\Omega(t-t_0)]\underline{G}(\underline{\tilde{\eta}},\underline{\tilde{\xi}}). \tag{33b}$$

Here the fact that the quantities $\underline{F}(\underline{\tilde{\eta}},\underline{\tilde{\xi}})$ and $\underline{G}(\underline{\tilde{\eta}},\underline{\tilde{\xi}})$ do not depend on Θ plays a crucial role, since it is then clear that this closed system of ODEs entails a *periodic* evolution—with period $T = 2\pi/\Omega$—of the dependent variables $\underline{\tilde{\xi}}(t)$ and $\underline{\tilde{\eta}}(t)$, as demonstrated via the usual trick to go over, from the independent variable t, to the 'periodic' timelike variable $\tilde{t}$ defined now, say, as follows:

$$\tilde{t}(t) = \frac{\sin[\Omega(t-t_0)]}{\Omega}, \qquad \dot{\tilde{t}} = \cos[\Omega(t-t_0)]. \tag{34}$$

The only possible difficulty arises if the specific definition of (one or both of) the collective variables H and Θ turns out to be incompatible with their periodic evolution (32), or likewise if the periodic evolutions of the coordinates $\underline{\tilde{\xi}}$ and $\underline{\tilde{\eta}}$ yield an unacceptable evolution of the original canonical variables $\underline{q}$ and $\underline{p}$. The fact that the first of these possibilities may indeed occur is demonstrated by the example detailed above; see (26*a*) and the treatment following this equation. As for the second possible difficulty, let us point out that the one-to-one character of the transformation between the variables $\underline{\tilde{p}}, \underline{\tilde{q}}$ and the variables $\underline{\tilde{\eta}}, \underline{\tilde{\xi}}$ (note that here the superimposed tildes are unessential, since they only refer to the assumed time evolution of these variables) implies that this difficulty can only emerge due to singularities in the time evolution of the system obtained from (33) via the change of the independent variable (34).

In any case this requires, as already indicated above, that some checks be made on a case-by-case basis. But, aside for the possible emergence of these difficulties, the *entirely isochronous* character of the motions yielded by the Ω-modified Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$, see (25*c*), has now been demonstrated.

2.6. A simple example

In this subsection we display, with minimal comments, the findings obtained by applying our Ω-modification technique to the very simple Hamiltonian describing a couple of equal mass one-dimensional particles interacting pairwise with a force proportional to their mutual distance; when this force is attractive this model corresponds of course, in the center-of-mass system, to the standard 'harmonic' oscillator model. (We write 'harmonic' under inverted commas to emphasize that the *entirely isochronous* Ω-modified Hamiltonians yielded by our technique *all* yield motions deserving to be called *harmonic*, inasmuch as they are characterized by just a single frequency of oscillation; in the case of many-body problems, the term 'nonlinear harmonic oscillators' [7] is perhaps the most appropriate to describe the corresponding dynamics . . .)

The original Hamiltonian:

$$H(p_1, p_2; q_1, q_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{\omega^2}{4}(q_1 - q_2)^2. \tag{35}$$

Some standard definitions and related formulae:

$$Q = \frac{q_1 + q_2}{2}, \qquad P = p_1 + p_2; \tag{36}$$

$$x_1 = q_1 - Q = \frac{q_1 - q_2}{2}, \qquad x_2 = q_2 - Q = \frac{q_2 - q_1}{2}, \qquad x_1 + x_2 = 0, \tag{37a}$$

$$y_1 = p_1 - \frac{P}{2} = \frac{p_1 - p_2}{2}, \qquad y_2 = p_2 - \frac{P}{2} = \frac{p_2 - p_1}{2}, \qquad y_1 + y_2 = 0; \tag{37b}$$

$$h(y_1, y_2; x_1, x_2) = \frac{1}{2}\left(y_1^2 + y_2^2\right) + \frac{\omega^2}{4}(x_1 - x_2)^2, \tag{38a}$$

$$H(p_1, p_2; q_1, q_2) = \frac{1}{4}P^2 + h(y_1, y_2; x_1, x_2); \tag{38b}$$

$$Q' = \frac{1}{2}P, \qquad P' = 0, \tag{39a}$$

$$Q(\tau) = Q(\tau_0) + \frac{1}{2}P(\tau_0)(\tau - \tau_0), \qquad P(\tau) = P(\tau_0); \tag{39b}$$

$$x_n' = y_n, \qquad y_n' = -\frac{\omega^2}{2}(x_n - x_{n+1}), \qquad n = 1, 2 \bmod(2), \tag{40a}$$

$$x_n(\tau) = x_n(\tau_0)\cos[\omega(\tau-\tau_0)] + y_n(\tau_0)\frac{\sin[\omega(\tau-\tau_0)]}{\omega}, \qquad n=1,2, \tag{40b}$$

$$y_n(\tau) = y_n(\tau_0)\cos[\omega(\tau-\tau_0)] - \omega x_n(\tau_0)\sin[\omega(\tau-\tau_0)], \qquad n=1,2. \tag{40c}$$

The Ω*-modified Hamiltonian*:

$$\tilde{H}(\tilde{p}_1,\tilde{p}_2;\tilde{q}_1,\tilde{q}_2;\Omega) = \frac{1}{2}\left\{\left[\tilde{P} + \frac{\tilde{h}(\tilde{y}_1,\tilde{y}_2;\tilde{x}_1,\tilde{x}_2)}{b}\right]^2 + \Omega^2\tilde{Q}^2\right\}; \tag{41}$$

$$\tilde{Q} = \frac{\tilde{q}_1+\tilde{q}_2}{2}, \qquad \tilde{P} = \tilde{p}_1+\tilde{p}_2; \tag{42}$$

$$\tilde{x}_1 = \tilde{q}_1 - \tilde{Q} = \frac{\tilde{q}_1-\tilde{q}_2}{2}, \qquad \tilde{x}_2 = \tilde{q}_2 - \tilde{Q} = \frac{\tilde{q}_2-\tilde{q}_1}{2}, \qquad \tilde{x}_1+\tilde{x}_2 = 0, \tag{43a}$$

$$\tilde{y}_1 = \tilde{p}_1 - \frac{\tilde{P}}{2} = \frac{\tilde{p}_1-\tilde{p}_2}{2}, \qquad \tilde{y}_2 = \tilde{p}_2 - \frac{\tilde{P}}{2} = \frac{\tilde{p}_2-\tilde{p}_1}{2}, \qquad \tilde{y}_1+\tilde{y}_2 = 0; \tag{43b}$$

$$\tilde{h}(\tilde{y}_1,\tilde{y}_2;\tilde{x}_1,\tilde{x}_2) = \frac{1}{2}\left(\tilde{y}_1^2+\tilde{y}_2^2\right) + \frac{\omega^2}{4}(\tilde{x}_1-\tilde{x}_2)^2; \tag{44a}$$

$$\begin{aligned}\tilde{H}(\tilde{p}_1,\tilde{p}_2;\tilde{q}_1,\tilde{q}_2;\Omega) = \frac{1}{2}\Bigg\{&\left[\tilde{p}_1+\tilde{p}_2+\frac{(\tilde{p}_1-\tilde{p}_2)^2}{4b}\right]^2 \\ &+ \frac{\omega^2}{2b}\left[\tilde{p}_1+\tilde{p}_2+\frac{(\tilde{p}_1-\tilde{p}_2)^2}{4b}\right](\tilde{q}_1-\tilde{q}_2)^2 + \left(\frac{\omega^2}{4b}\right)^2(\tilde{q}_1-\tilde{q}_2)^4 + \frac{\Omega^2}{4}(\tilde{q}_1+\tilde{q}_2)^2\Bigg\}.\end{aligned} \tag{45}$$

The Ω*-modified equations of motions*: see (16) and (19) yielding

$$\dot{\tilde{x}}_n = \frac{1}{b}\left[\tilde{P} + \frac{\tilde{h}(\tilde{y}_1,\tilde{y}_2;\tilde{x}_1,\tilde{x}_2)}{b}\right]\tilde{y}_n, \qquad n=1,2, \tag{46a}$$

$$\dot{\tilde{y}}_n = -\frac{\omega^2}{2b}\left[\tilde{P} + \frac{\tilde{h}(\tilde{y}_1,\tilde{y}_2;\tilde{x}_1,\tilde{x}_2)}{b}\right](\tilde{x}_n-\tilde{x}_n), \qquad n=1,2\mathrm{mod}(2). \tag{46b}$$

The isochronous *motions yielded by the* Ω*-modified Hamiltonian* $\tilde{H}(\tilde{p}_1,\tilde{p}_2;\tilde{q}_1,\tilde{q}_2;\Omega)$: see (17), and (21) yielding (40*b*) and (40*c*), namely (see (21*b*))

$$\begin{aligned}\tilde{x}_n(t) = \tilde{x}_n(0)\cos&\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\} \\ &+ \frac{\tilde{y}_n(0)}{\omega}\sin\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\}, \qquad n=1,2,\end{aligned} \tag{47a}$$

$$\begin{aligned}\tilde{y}_n(t) = \tilde{y}_n(0)\cos&\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\} \\ &- \omega\tilde{x}_n(0)\sin\left\{\omega C\frac{\sin[\Omega(t-t_0)]+\sin(\Omega t_0)}{\Omega}\right\}, \qquad n=1,2,\end{aligned} \tag{47b}$$

with C and t_0 defined in terms of the initial data by (20*c*).

The *entirely isochronous* character of this motion, with period $T = 2\pi/\Omega$, is evident (see (43*a*), (17) and (47)), and note that this outcome obtains even if ω is purely imaginary, $\omega = \mathrm{i}\alpha$ with α real, in which case the original Hamiltonian H and the Ω-modified Hamiltonian $\tilde{H}$, and as well of course the corresponding solutions, see (43*a*), (17) and (47), are nevertheless all *real*.

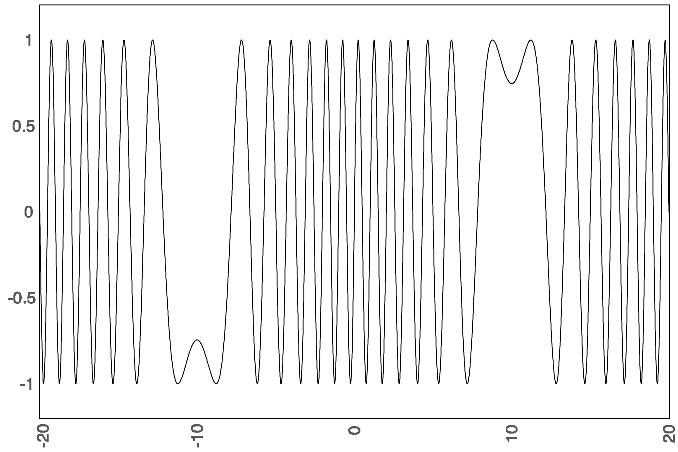

Figure 1. Graph (over one time period) of $\tilde{x}_n(t)$, see (47*a*), for $\tilde{x}_n(0) = 0$, $\tilde{y}_n(0) = \omega = 40\Omega = 2\pi$, $\Omega = \pi/20$, $C = 1$, $t_0 = 0$. Note the overall periodicity with period $T = 2\pi/\Omega = 40$, the large regions where the behavior is nearly periodic with the original period $2\pi/\omega = 1$ of the solutions of the unmodified Hamiltonian (35), and the transition regions around the times $t = 10$ and $t = 30$ when $\dot{\tilde{t}}(t)$ vanishes.

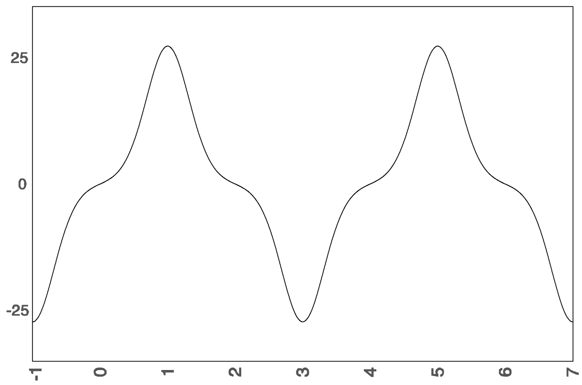

Figure 2. Graph (over two time periods) of $\tilde{x}_n(t)$, see (47*a*), for $\tilde{x}_n(0) = 0$, $\tilde{y}_n(0) = 2\pi$, $\omega = 4\mathrm{i}\Omega = 2\pi\,\mathrm{i}$, $\Omega = \pi/2$, $C = 1$, $t_0 = 0$. Note the overall periodicity with period $T = 2\pi/\Omega = 4$, the regions where the time evolution resembles the original $\sin(\omega t)/i = \sinh(2\pi t)$ behavior of the corresponding solution of the unmodified Hamiltonian (35), and the transition regions around the times $t = -1, 1, 3, 5, 7$ when $\dot{\tilde{t}}(t)$ vanishes.

3. The quantum case

In this section we tersely show why, in a *quantal* context, the Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$, see (13), features an (infinitely degenerate) *equispaced* spectrum with spacing $\hbar\Omega$.

This spectrum consists of the eigenvalues E_k of the stationary Schrödinger equation:

$$\frac{1}{2}\left\{\left[-\mathrm{i}\hbar\frac{\partial}{\partial Z} + \frac{\lambda}{b}\right]^2 + \Omega^2 Z^2\right\}\psi_k(Z;\lambda) = E_k\psi_k(Z;\lambda), \tag{48a}$$

obtained from expression (13) of the Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$ via the standard quantization rule

$$\tilde{P} \Longrightarrow -\mathrm{i}\hbar\frac{\partial}{\partial Z}, \qquad \tilde{Q} \Longrightarrow Z, \tag{48b}$$

and by identifying λ as an eigenvalue of the quantized version of the relative-motion Hamiltonian $\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})$, see (9a). Indeed this Schrödinger equation is obtained by assuming the eigenfunctions of the quantized version of the Hamiltonian $\tilde{H}(\underline{\tilde{p}}, \underline{\tilde{q}}; \Omega)$ factor into the product of an eigenfunction, $\psi_k(Z;\lambda)$, depending on the variable Z and on which acts the differential operator $\partial/\partial Z$ (see (48a)), and of the eigenfunction corresponding to the eigenvalue λ of the quantized version of the relative-motion Hamiltonian $\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})$. The justification for this factorization is in the commutativity of the operators representing the quantal versions of the canonical variables $\tilde{P}$ and $\tilde{Q}$ with the operator representing the quantal version of the relative-motion Hamiltonian $\tilde{h}(\underline{\tilde{y}}, \underline{\tilde{x}})$—a commutativity reflecting the Poisson commutativity of the corresponding quantities in the classical context, see (18). It is now plain that the Schrödinger equation (48a) features the spectrum

$$E_k = \hbar\Omega\left(k + \tfrac{1}{2}\right), \qquad k = 0, 1, 2, \ldots, \tag{49a}$$

with the corresponding eigenfunctions reading

$$\psi_k(Z;\lambda) = \exp\left(\frac{\mathrm{i}\lambda z}{b\sqrt{\hbar\Omega}} - \frac{z^2}{2}\right) H_k(z), \qquad z = Z\sqrt{\frac{\Omega}{\hbar}}, \tag{49b}$$

where $H_k(z)$ denotes the standard Hermite polynomial of order k.

This spectrum, (49a), is of course *equispaced* with spacing $\hbar\Omega$, and it is *infinitely degenerate* inasmuch as it does not feature any dependence on the eigenvalues λ.

4. Outlook

It has been said that scientific progress consists in turning surprising findings into obvious truths. The results presented in this paper might be seen as belonging to this pattern. It was indeed our experience when describing these findings to competent colleagues to be initially met by widespread skepticism: that such a vast class of (*autonomous*) Hamiltonians could so readily and generally be Ω-modified obtaining thereby new (*autonomous*) Hamiltonians yielding *entirely isochronous* motions appeared queer indeed hard to believe, even more so when the original Hamiltonians are known to yield *chaotic* motions; hence the corresponding Ω-modified Hamiltonians should feature the phenomenology of *transient chaos*, as outlined above. But of course once the mechanism underlying all this was understood and internalized—namely, the fact that, via a tricky transformation, time itself was effectively made *periodic*—the origin of this remarkable phenomenology became transparent and its manifestations essentially obvious. We nevertheless deem that these findings deserve further investigation; in particular we soon plan to exhibit further instances of this phenomenology for systems of several particles interacting via 'realistic' potentials, when issues of reversibility become important. We also plan to investigate the statistical mechanics and thermodynamics of macroscopic systems described by Ω-modified Hamiltonians yielding *entirely isochronous* motions, while we will instead refrain from proposing pseudo-philosophical interpretations of our Ω-modified Hamiltonians as describing ever-recurring universes (but let us not forget that metrics yielding closed timelike geodesics are indeed predicted in some black hole models, see for instance [8]).

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